

# LINEAR ALGEBRA AND THE ROTATION OF THE EARTH\*

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## I. Introduction.

At the beginning of this century, linear algebra had very few applications in mathematical physics. Even as late as 1926, when Heisenberg and Born introduced matrix methods into quantum mechanics, the fact that matrix multiplication is not commutative was regarded as one of the most bizarre aspects of the new theory. Today, linear algebra has penetrated virtually every branch of mathematical physics, from cosmology to elementary particle theory, and it would be quite impossible to give a useful survey of its manifold applications throughout the subject. Instead, these notes describe a typical physical problem in which linear algebra arises in a natural and striking way.

Broadly speaking, there are two ways in which linear algebra can be of importance in a physical theory. Firstly, linear algebra is of course a useful *technique*. For example, many physical theories involve large systems of linear equations, and linear algebra provides concepts and methods for solving such systems, or at least for deciding whether solutions exist. However, linear algebra can also arise in a second and much more fundamental way. We are familiar with the idea that the basic

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quantities of a physical theory (such as Newtonian mechanics) can be either scalars or vectors. In such a theory, linear algebra is not merely a useful tool : rather, it forms an essential part of the theory itself. The theory of rotating rigid bodies gives an interesting example of this second type of application of linear algebra, and forms the subject of these notes.

## II. Rotation of Rigid Bodies in Two Dimensions

A rigid body can be defined as a system of particles such that the distance between any two particles is independent of time. In this section we consider such a body rotating about a fixed axis A. In this case the path of each particle is confined to a plane, so the whole problem is essentially two-dimensional.

Let  $m_i$  be the mass of the  $i$ th particle, and  $r_i$  be its perpendicular distance from the axis A. (For the present, we treat the body as a finite collection of particles.) The *moment of inertia* of the particle is defined as  $m_i r_i^2$ . The moment of inertia of the whole body is defined as  $\sum m_i r_i^2$ , where the sum will always be taken over the whole set of particles. (For a continuous body, we replace  $\sum$  by  $\int$  in the usual way, and obtain  $\int r^2 dm$ .) Clearly the moment of inertia of a body depends on its mass, shape, and the distribution of mass within it. It also depends on the orientation of the axis A.

If  $V_i$  is the speed of the  $i$ th particle, then the total kinetic energy (K.E.) of the body is  $\sum \frac{1}{2} m_i V_i^2$ . Now if  $\omega$  is the angular speed of rotation of the body, we obviously have  $V_i = \omega r_i$ , and so  $K.E. = \sum \frac{1}{2} m_i V_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega^2$ , whence  $K.E. = \frac{1}{2} I \omega^2$ ,

where  $I = \sum m_i r_i^2$  is the moment of inertia. Notice the analogy of this with the formula  $K.E. = \frac{1}{2} MV^2$ .

The *angular momentum* of the  $i$ th particle is defined as  $m_i r_i v_i$ . The total angular momentum is  $h = \sum m_i r_i v_i = \sum m_i r_i^2 \omega = I\omega$ . Notice the analogy with the usual formula for momentum. The principle of angular momentum *conservation* states that a system on which no external torques act has constant angular momentum. (Here we can think of "torques" as "twisting forces".)

Clearly, all of the quantities describing the rotational motion of a rigid body about a fixed axis (kinetic energy, angular momentum, etc.) are controlled by  $I$ , the moment of inertia. In any given problem, we have to calculate  $I$  from the shape and mass-distribution of the body; in principle, this is just an exercise in integral calculus, using  $I = \int r^2 dm$ .

### III. Rotational Motion in 3 Dimensions

The study of rotational motion about a fixed axis is somewhat artificial, because (unless it is forced to do so) an arbitrary rigid body will not rotate steadily about an axis which is fixed in space. Rather, it will tend to tumble around in a complicated way, and we can only speak of an *instantaneous* axis of rotation. In order to deal with this, we need some way of describing rotations in 3 dimensions.

By its very nature, a rotation does not change the lengths of vectors or the angles between them. Any rotation can therefore be represented by a linear transformation  $R$  which is orthogonal, i.e. if  $I$  denote the matrix of  $R$  with respect to some basis (say

$i, j, k$ ) by  $[R]$ , then  $[R]^T[R] = 3 \times 3$  identity matrix. (Henceforth we adopt the custom of using the same symbol for a linear transformation and its matrix with respect to a given basis, i.e. we drop the  $[ \ ]$ .)

We shall describe the position of a rotating rigid body as follows. At the centre of rotation, set up an  $i, j, k$  basis, fixed in space as usual. Now imagine an orthonormal basis  $\vec{l}, \vec{m}, \vec{n}$  embedded in the body and rotating with it. We choose  $\vec{l}, \vec{m}, \vec{n}$ , such that, at time  $t = 0$ ,  $\vec{l} = i, \vec{m} = j, \vec{n} = k$ . Then as time goes on,  $\vec{l}, \vec{m}$ , and  $\vec{n}$  rotate away from  $i, j, k$ . By comparing the two bases, we can tell how far the body has rotated. Of course, we will have  $\vec{l} = Ri, \vec{m} = Rj, \vec{n} = Rk$ , where  $R$  is a rotation transformation which *must depend on time*, since  $\vec{l}, \vec{m}, \vec{n}$  are changing while  $i, j, k$  are fixed. That is, the matrix of  $R$  will have components which are *functions of time* instead of numbers

More generally, let  $\vec{r}(t)$  be the position vector of any particle in the body. Then we have  $\vec{r}(t) = R\vec{r}(0)$ , where  $\vec{r}(0)$  is the position of the particle at time  $t = 0$  (and is therefore a constant vector.) The velocity  $\vec{v}$  is  $\vec{v} = \frac{d}{dt} \vec{r} = \left(\frac{dR}{dt}\right) \vec{r}(0)$ . Now since  $R^T$  is the inverse matrix of  $R$ , we have  $\vec{r}(0) = R^{-1} \vec{r}(t) = R^{T\vec{r}}(t)$ , and so, substituting this into the formula for  $\vec{v}$ , we get  $\vec{v} = \left(\frac{dR}{dt}\right) R^T \vec{r}(t) = S\vec{r}(t)$ , where by definition  $S = \left(\frac{dR}{dt}\right) R^T$  (matrix product). Thus we see that the velocity of a particle in a rigid body is related to its position vector by the linear transformation  $S$ .

Now  $S$  has the crucial property of being *antisymmetric*. That is,  $S^T = -S$ . To see this, take the equation  $RR^T = \text{identity}$  and differentiate both sides. Then we get  $\frac{dR}{dt} R^T + R \frac{d}{dt} (R^T) = 0$ . The

first term is  $S$ , and the second is  $S^T$ . To see this, remember that for any two matrices  $A, B$  we have  $(AB)^T = B^T A^T$ . Hence

$$S^T = \left(\frac{dR}{dt} R^T\right)^T = (R^T)^T \left(\frac{dR}{dt}\right)^T = R \frac{dR^T}{dt}$$

Thus we have  $S + S^T = 0$ , i.e.  $S^T = -S$ . Now let us consider the matrix  $S$ . It is easy to see that any  $3 \times 3$  antisymmetric matrix

must have the form  $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ . Thus,  $S$  must have this form.

Writing  $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$  and  $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  we have, from  $\vec{v} = S\vec{r}$ ,

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ay + bz \\ -ax + cz \\ bx - cy \end{bmatrix}$$

But this answer looks suspiciously like a vector (cross) product of two vectors. In fact, some experimentation shows that if we

define  $\vec{\omega} = \begin{bmatrix} -c \\ b \\ -a \end{bmatrix}$ , then  $\vec{\omega} \times \vec{r} = \begin{bmatrix} ay + bz \\ -ax + cz \\ -bx - cy \end{bmatrix}$ .

Thus we have the important formula

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

But what is  $\vec{\omega}$ ? Suppose  $\vec{r}$  and  $\vec{v}$  both lie in the  $xy$  plane. It is easy to see that  $\vec{\omega}$  must point along the  $z$  axis, so we can write it as  $\vec{\omega} = \omega k$ . Then one finds easily that  $|\vec{v}| = \omega |\vec{r}|$ , i.e.  $\omega$  is just the angular speed. Thus  $\vec{\omega}$  is a 3-dimensional, vectorial generalisation of angular speed. We call  $\vec{\omega}$  the *angular velocity*

vector. It is possible to show that the *magnitude* of  $\vec{\omega}$  gives the angular speed about the instantaneous axis of rotation, whereas the *direction* of  $\vec{\omega}$  gives the direction of the instantaneous axis. Thus  $\vec{\omega}$  is not usually constant; as the body tumbles around,  $\vec{\omega}$  changes so that it always points along the instantaneous axis.

The above derivation of the formula  $\vec{V} = \vec{\omega} \times \vec{r}$  provides an example of the first type of application of linear algebra mentioned in the introduction. Notice that there are no matrices in the final result; we only used them as a technical aid, and, in fact, there are other derivations of this result which make no mention of matrices whatever. We now turn to an application to the second type.

#### IV. Moment of Inertia as a Linear Transformation

In 3 dimensions, we define the angular momentum of the  $i$ th particle as  $\vec{r}_i \times m_i \vec{v}_i$  (vector product), and the total angular momentum as  $\sum \vec{r}_i \times m_i \vec{v}_i = \vec{h}$ . (It is not difficult to show that these definitions reduce to the previous ones if we confine everything to the xy plane.) Now for each particle, we have  $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ , and so  $\vec{h} = \sum m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$ . Now using the identity  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ , we obtain 
$$h = \sum m_i (\vec{r}_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i).$$

Notice that  $\vec{h}$  is *not* proportional to  $\vec{\omega}$ , i.e. there does not necessarily exist any number  $c$  with  $\vec{h} = c\vec{\omega}$ . So how is  $\vec{h}$  obtained from  $\vec{\omega}$ ? The map

$$\vec{\omega} \longrightarrow \sum m_i (\vec{r}_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i)$$

is, in fact, simply a linear transformation, as is easily verified. That is, we have the equation

$$\vec{h} = I\vec{\omega} ,$$

where  $I$  (not to be confused with the identity matrix) is not a number, but rather a linear transformation. Comparing this with the two-dimensional formula  $h = I\omega$ , we see that the linear transformation  $I$  takes the place of the moment of inertia. Thus, in three dimensions, the moment of inertia is no longer a scalar, but nor is it a vector — rather, it becomes a linear transformation, which we call the *moment of inertia transformation*.

This is an example of the second type of application of linear algebra in physics; here, the moment of inertia transformation is a basic physical quantity like angular velocity or angular momentum. It has not been introduced as a mere technical convenience.

For a continuous distribution of matter, we replace the  $\sum$  by  $\int$  and obtain

$$\vec{h} = I\vec{\omega} \left( \int \vec{r}^2 dm \right) \vec{\omega} - \int (\vec{r} \cdot \vec{\omega}) \vec{r} dm.$$

By letting  $I$  act on  $i, j, k$  we can work out the matrix of  $I$  with respect to the  $i, j, k$  basis. Setting  $\vec{r} = xi + yj + zk$  as usual, we find, for example

$$\begin{aligned} Ii &= \left( \int (x^2 + y^2 + z^2) dm \right) i - \int (x)(xi + yj + zk) dm \\ &= \int (y^2 + z^2) dm i - \int xy dm j - \int xz dm k . \end{aligned}$$

Proceeding in this way, we find that the matrix of  $I$  in the  $i j k$  basis must be

$$\begin{bmatrix} \int (y^2 + z^2) dm & -\int xy dx & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{bmatrix}$$

Thus,  $I$  can be computed given the shape and mass distribution of the body.

Notice that the matrix of  $I$  is *symmetric*, that is, equal to its own transpose. This is of great importance when we consider the eigenvalue problem for  $I$ , to which we now turn.

Since  $I$  has a symmetric  $3 \times 3$  matrix, we can find 3 orthogonal eigenvectors with corresponding eigenvalues  $I_1, I_2, I_3$ . When

referred to the eigenvector basis,  $I$  has matrix 
$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

What is the physical meaning of these eigenvectors and eigenvalues - if any? Suppose we set the body spinning about the first eigenvector. Since  $\vec{\omega}$  points in the direction of the axis of rotation, this means that we are taking  $\vec{\omega}$  to be the first eigenvector.

Thus we have the eigenvalue equation

$$I\vec{\omega} = I_1\vec{\omega}$$

But  $\vec{h} = I\vec{\omega}$ , so  $\vec{h} = I_1\vec{\omega}$ . Taking the magnitude of these vectors, we first  $|\vec{h}| = |I_1||\vec{\omega}|$ . Comparing this with the two-dimensional equation  $h = I\omega$ , we see that in fact  $I_1$  is simply the ordinary

(scalar) moment of inertia about the direction of the first eigenvector. Similarly for  $I_2$  and  $I_3$  : *the physical meaning of the eigenvalues is that they are the moments of inertia about axes parallel to the eigenvectors.*

But what of the eigenvectors themselves? Suppose no torque acts on the body. Then, by the conservation of angular momentum,  $\vec{h}$ , the direction of  $\vec{\omega}$  is constant. This means that the object is rotating steadily about a fixed axis in space, and not tumbling around as it usually does. Thus, *the physical meaning of the eigenvectors of  $I$  is that they give the directions in the body about which steady rotation (without tumbling) is possible.* Intuitively, the reason for this is that the eigenvectors point along any axes of symmetry in the body.

## V. The Euler Equations

Let us adopt a basis of orthonormal eigenvectors of  $I$ . Since it is possible for the body to rotate steadily about them, they must be fixed in the body like the vectors  $\vec{l}$ ,  $\vec{m}$ ,  $\vec{n}$  discussed in section III. Hence we shall take  $\vec{l}$ ,  $\vec{m}$ ,  $\vec{n}$  to be unit eigenvectors of  $I$ . Let  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  be the components of  $\vec{\omega}$  with respect to this basis. Then the angular momentum vector  $\vec{h}$  can be written as

$$\begin{aligned} \vec{h} = I\vec{\omega} &= I(\omega_1\vec{l} + \omega_2\vec{m} + \omega_3\vec{n}) \\ &= \omega_1 I\vec{l} + \omega_2 I\vec{m} + \omega_3 I\vec{n} && \text{(linearly)} \\ &= \omega_1 I_1\vec{l} + \omega_2 I_2\vec{m} + \omega_3 I_3\vec{n} && \text{(eigenvectors)}. \end{aligned}$$

Now suppose that no external torques act. Then by the conservation of angular momentum,

$$0 = \frac{d}{dt} \vec{h} = I_1 \frac{d}{dt} (\omega_1 \vec{l}) + I_2 \frac{d}{dt} (\omega_2 \vec{m}) + I_3 \frac{d}{dt} (\omega_3 \vec{n}).$$

since  $I_1, I_2, I_3$  are constants. (They are moments of inertia of a rigid body about definite axes.) Of course,  $\omega_1, \omega_2, \omega_3$  will not usually be constants, and nor are  $\vec{l}, \vec{m}, \vec{n}$ , because they are rotating with the body and are therefore functions of time. But by our general formula

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r},$$

we have

$$\frac{d\vec{l}}{dt} = \vec{\omega} \times \vec{l}, \quad \frac{d\vec{m}}{dt} = \vec{\omega} \times \vec{m}, \quad \frac{d\vec{n}}{dt} = \vec{\omega} \times \vec{n}.$$

So

$$0 = I_1 \frac{d\omega_1}{dt} \vec{l} + I_1 \omega_1 \vec{\omega} \times \vec{l} + I_2 \frac{d\omega_2}{dt} \vec{m} + I_2 \omega_2 \vec{\omega} \times \vec{m} \\ + I_3 \frac{d\omega_3}{dt} \vec{n} + I_3 \omega_3 \vec{\omega} \times \vec{n}.$$

But  $\vec{\omega} \times \vec{l} = (\omega_1 \vec{l} + \omega_2 \vec{m} + \omega_3 \vec{n}) \times \vec{l}$ , which we can work out by using  $\vec{l} \times \vec{l} = 0, \vec{l} \times \vec{m} = \vec{n}, \vec{m} \times \vec{n} = \vec{l}, \vec{n} \times \vec{l} = \vec{m}$  (by analogy with  $\mathbf{i} \times \mathbf{i} = 0, \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$ ). So we get

$$\vec{\omega} \times \vec{l} = -\omega_2 \vec{n} + \omega_3 \vec{m}$$

$$\vec{\omega} \times \vec{m} = \omega_1 \vec{n} - \omega_3 \vec{l}$$

$$\vec{\omega} \times \vec{n} = \omega_2 \vec{l} - \omega_1 \vec{m}$$

Substituting these into the above, we obtain

$$\begin{aligned}
 0 = & \left( \frac{d\omega_1}{dt} + I_3\omega_3\omega_2 - I_2\omega_2\omega_3 \right) \vec{e}_1 \\
 & + \left( I_2 \frac{d\omega_2}{dt} + I_1\omega_1\omega_3 - I_3\omega_3\omega_1 \right) \vec{e}_2 \\
 & + \left( I_3 \frac{d\omega_3}{dt} + I_2\omega_2\omega_1 - I_1\omega_1\omega_2 \right) \vec{e}_3,
 \end{aligned}$$

which implies the three scalar equations

$$\begin{aligned}
 0 = & I_1 \frac{d\omega_1}{dt} + I_3\omega_3\omega_2 - I_2\omega_2\omega_3 \\
 0 = & I_2 \frac{d\omega_2}{dt} + I_1\omega_1\omega_3 - I_3\omega_3\omega_1 \\
 0 = & I_3 \frac{d\omega_3}{dt} + I_2\omega_2\omega_1 - I_1\omega_1\omega_2.
 \end{aligned}$$

These are the equations of motion of a rotating rigid body on which no external torques act. They are known as the Euler equations.

## VI. The Rotation of the Earth

In any particular situation, we have to calculate  $I_1$ ,  $I_2$ ,  $I_3$  from the shape and mass distribution of the body in question, and then substitute them into the Euler equations and solve for  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$ . These will tell us  $\vec{\omega}$ , that is, the axis or rotation, as a function of time. We now apply this to the rotation of the Earth.

To a very good approximation, the Earth is an ellipsoid of revolution, being slightly flattened at the poles. If  $a$  and  $b$

are the distances from the centre to the pole and to the equator respectively, then putting the polar axis along the z axis, we have

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$$

as the equation of the surface of the Earth. For simplicity, we make the assumption that the Earth has constant density  $\rho$ . Of course, this is not a very good approximation, but we shall return to this later. We now have  $dm = \rho dV$ , where  $dV$  is the element of volume. The matrix of  $I$  now becomes

$$\begin{bmatrix} \rho \iiint (y^2 + z^2) dx dy dz, & -\rho \iiint xy dx dy dz, & -\rho \iiint xz dx dy dz \\ -\rho \iiint xy dx dy dz, & \rho \iiint (x^2 + z^2) dx dy dz, & -\rho \iiint yz dx dy dz \\ -\rho \iiint xz dx dy dz, & -\rho \iiint yz dx dy dz, & \rho \iiint (x^2 + y^2) dx dy dz \end{bmatrix}$$

Evaluating the triple integrals, we find that the matrix becomes (if  $M =$  mass of the Earth)

$$\begin{bmatrix} \frac{1}{5} M (a^2 + b^2) & 0 & 0 \\ 0 & \frac{1}{5} M (a^2 + b^2) & 0 \\ 0 & 0 & \frac{2}{5} Mb^2 \end{bmatrix}$$

(Here we use the fact that the volume of the ellipsoid is  $\frac{4}{3} \pi ab^2$ , so  $\rho = M/\frac{4}{3} \pi ab^2$ ). Hence we can read off the eigenvalues directly: they are  $I_1 = I_2 = \frac{1}{5} M (a^2 + b^2)$ ,  $I_3 = \frac{2}{5} Mb^2$ .

Substituting into the Euler equations, we find

$$0 = \frac{d\omega_1}{dt} + \frac{I_3 - I_1}{I_1} \omega_2 \omega_3$$

$$0 = \frac{d\omega_2}{dt} + \frac{I_1 - I_3}{I_1} \omega_1 \omega_3$$

$$0 = \frac{d\omega_3}{dt} \text{ since } I_1 = I_2 .$$

If we define  $K = \frac{I_3 - I_1}{I_1} \omega_3$ , then  $K$  is a constant (since  $\omega_3$  is constant) which is given by  $K = \left( \frac{b^2 - a^2}{b^2 + a^2} \right) \omega_3$ . This first two

Euler equations become

$$\frac{d\omega_1}{dt} = -K\omega_2$$

$$\frac{d\omega_2}{dt} = K\omega_1$$

Differentiating the first equation and using the second, we get

$$\frac{d^2\omega_1}{dt^2} = -K \frac{d\omega_2}{dt} = -K^2 \omega_1$$

which is the equation satisfied by  $c \cos (Kt)$ , where  $c$  is any constant. Then from the first equation  $\omega_2 = \frac{d\omega_1}{dt} / K = c \sin (Kt)$ . Thus we find that

$$\omega_1 = c \cos Kt, \omega_2 = c \sin Kt, \omega_3 = \text{constant}.$$

Note that  $\vec{\omega}^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = c^2 + \omega_3^2$  which is constant. The equations for  $\omega_1$  and  $\omega_2$  are the parametric equations of a circle. Thus we can imagine  $\vec{\omega}$  as a vector of constant length

whose tip sweeps out a circle as time goes by. It will complete

an entire circle in a time  $2\pi/K = \frac{2\pi(a^2 + b^2)}{\omega_3(b^2 - a^2)} = T$ .

Now recall that the direction of  $\vec{\omega}$  gives the instantaneous axis of rotation. Thus an observer fixed on the surface of the Earth should see the axis of rotation of the Earth moving in a circle about the symmetry axis. The period of this motion is given by the above formula for  $T$ . The polar diameter of the Earth is about 12,640 kilometres, and its equatorial diameter about 12,680 kilometres. Hence  $a = 6320$ ,  $b = 6340$ , and  $\omega_3$  is, to a good approximation, the angular speed of the Earth, that is,  $2\pi/1$  day. We find  $T = 306$  days.

Let us summarize. The rotational motion of the Earth is describe by is angular velocity vector  $\vec{\omega}$ , which is related to the angular momentum vector  $\vec{h}$  by a linear transformation,  $\vec{h} = I\vec{\omega}$ . The conservation of angular momentum (for a body on which no external torques act) leads us to the Euler equations, which can be solved for  $\vec{\omega}$  provided that we have the eigenvalues  $I_1, I_2, I_3$  of the linear transformation  $I$ . Applying this to the case of the Earth, we find that our theory predicts that the axis of rotation of the Earth should move in a circle around the geographical North pole with a period of about 300 days. We do not expect our numerical value to be very accurate, for the following reasons.

(i) We assumed that the Earth has the same density  $\rho$  throughout. This is, of course, not correct : the Earth consists of several distinct layers, of different densities. However, the situation here is not as unsatisfactory as it may appear. Note that it is the *ratio* of the quantities  $I_1$  and  $I_3$  which enters into the

expression for  $T$ , so that it seems reasonable to expect some cancellation of the errors here.

(ii) We assumed that no external torques act on the Earth. This is not actually valid, but it can be shown that this is a source of rather small errors.

(iii) The most serious source of error, perhaps surprisingly, is the assumption that the Earth is rigid. In fact, the Earth has a partly elastic interior, which allows it to dissipate energy through frictional losses, and to gain energy through (for example) earthquakes.

These remarks do not mean that our discussion of the rotation of the Earth is of no value. Obviously the Earth is an extremely complicated system which we can only describe by starting with a simple model which we gradually improve. In fact, a small effect of the type predicted by our theory (i.e. a "wandering" of the axis of rotation about the North-South axis) has been observed. As expected, the actual period is not exactly 300 days; it is closer to 400 days. In view of the approximations made, this is not a serious discrepancy. This motion is called the Earth's "*Chandler wobble*". The precise nature of the Chandler wobble is still being actively studied today; see the references below, or any textbook on geophysics.

## VII. Conclusion

The Chandler wobble is one of the most important aspects of the Earth's rotation. Its study provides a very concrete application of linear algebra. Notice that if the angular

momentum vector  $\vec{h}$  were simply *proportional* to  $\vec{\omega}$ , as it is in two dimensions, then there could be no Chandler wobble, because (by the conservation of angular momentum)  $\vec{h}$  points in a fixed direction. It is only because  $I$  is a *linear transformation* and not a number that it is possible for  $\vec{h}$  and  $\vec{\omega}$  to point in completely different directions ( $\vec{h} = I\vec{\omega}$ ), and this is what makes the Chandler wobble possible.

Thus we see that the viewpoint of linear algebra helps to clarify our understanding of this basic geophysical phenomenon.

#### Further Reading

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